

On self-dual MRD codes

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ABSTRACT. We investigate self-dual MRD codes. In particular we prove that a Gabidulin code in $(\mathbb{F}_q)^{n \times n}$ is equivalent to a self-dual code if and only if its dimension is $n^2/2$, $n \equiv 2 \pmod{4}$, and $q \equiv 3 \pmod{4}$. On the way we determine the full automorphism group of Gabidulin codes in $(\mathbb{F}_q)^{n \times n}$.

Keywords: self-dual MRD code, automorphism group, Gabidulin code

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1 Introduction.

Following Delsarte [2] a rank metric code is a set $\mathcal{C} \subseteq k^{m \times n}$ of $m \times n$ matrices over a field k . The distance between two matrices $A, B \in k^{m \times n}$ is defined as $d(A, B) := \text{Rk}(A - B)$, i.e. the rank of the difference of A and B . As usual we denote by

$$d(\mathcal{C}) := \min\{d(A, B) \mid A, B \in \mathcal{C}, A \neq B\}$$

the minimum distance of \mathcal{C} . The dual code of \mathcal{C} is

$$\mathcal{C}^\perp = \{X \in k^{m \times n} \mid (C, X) := \text{trace}(CX^\top) = 0 \text{ for all } C \in \mathcal{C}\}$$

where X^\top is the transpose and $\text{trace}(X)$ the trace of the matrix X . Clearly, \mathcal{C}^\perp is always a k -linear code, i.e. a subspace of the k -vector space $k^{m \times n}$.

Throughout the paper we assume that $m \geq n$, so our matrices have at least as many rows as columns. We will also assume that \mathcal{C} is a linear code. If $\mathcal{C} \leq k^{m \times n}$ has dimension ℓ and minimum distance d , then $d \leq n - \ell/m + 1$ (see [2, Theorem 5.4], [9, Theorem 8]). Codes where equality holds are called MRD codes (maximum rank distance codes). By [2, Theorem 5.4] the dual of an MRD code is again an MRD code (see also [9, Corollary 41]).

In this note we investigate self-dual MRD codes, i.e. MRD codes \mathcal{C} with $\mathcal{C} = \mathcal{C}^\perp$. As $\dim(\mathcal{C}) + \dim(\mathcal{C}^\perp) = \dim(k^{m \times n}) = mn$ a self-dual MRD code $\mathcal{C} \leq k^{m \times n}$ with $m \geq n$ has dimension $\frac{mn}{2}$ and minimum distance $d(\mathcal{C}) = \frac{n}{2} + 1$.

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Section 2 investigates which rank distance preserving linear automorphisms of $k^{m \times n}$ stabilise the inner product $(-, -)$ defined by $(A, B) = \text{trace}(AB^\top)$.

This inner product $(-, -)$ is the standard inner product if we identify $k^{m \times n}$ with $k^{1 \times mn}$. If $\text{char}(k) = 2$, then self-dual codes in $k^{1 \times mn}$ always contain the all-ones vector. So self-dual rank metric codes contain the all-ones matrix $J \in \{1\}^{m \times n}$ of rank 1. This implies that there are no self-dual MRD codes over fields of characteristic 2 (see Theorem 3.1). In Section 4 we give in odd characteristic a handy criterion to prove if a given rank metric code in $k^{m \times n}$ is equivalent to a self-dual code (see Theorem 4.3).

In the rest of the paper we study MRD codes in $k^{n \times n}$ where k is a finite field. In case $n = 2$ all self-dual MRD codes are classified in Section 3: They exist if and only if -1 is not a square in k .

The most well-studied examples of MRD codes are the Gabidulin codes ([3], [2]). Section 5 treats Gabidulin codes of full length n , i.e. $n = [K : k]$ is the degree of the field extension, as k -linear subspaces of dimension ℓn of $k^{n \times n}$. We determine the k -linear automorphism group of these codes (see Corollary 5.7) and show that such a Gabidulin code is equivalent to a self-dual code if and only if $n \equiv 2 \pmod{4}$, $\ell = n/2$, and -1 is not a square in k (see Theorem 5.10).

If -1 is a square in k or n is a multiple of 4, we do not have any examples of self-dual MRD codes in $k^{n \times n}$. Note that according to [7] there are 5 equivalence classes of self-dual MRD codes in $\mathbb{F}_5^{4 \times 2}$.

2 Automorphisms preserving the inner product.

The rank distance preserving automorphisms of $k^{m \times n}$ are

$$\kappa_{X,Y,Z,\sigma} : A \mapsto XA^\sigma Y + Z \text{ with } X \in \text{GL}_m(k), Y \in \text{GL}_n(k), Z \in k^{m \times n}, \sigma \in \text{Aut}(k)$$

or

$$\tau_{X,Y,Z,\sigma} : A \mapsto XA^{\top,\sigma} Y + Z \text{ with } X, Y \in \text{GL}_n(k), Z \in k^{m \times n}, \sigma \in \text{Aut}(k) \text{ (if } m = n\text{)}$$

and these are k -linear, if and only if $Z = 0$ and $\sigma = \text{id}$ (see [11], Theorem 3.4).

If $m = n$, then the $\tau_{X,Y} := \tau_{X,Y,0,\text{id}}$ are called **improper** and the $\kappa_{X,Y} := \kappa_{X,Y,0,\text{id}}$ **proper** automorphisms. The group of proper automorphisms is a normal subgroup of index 2 in the full automorphism group of $k^{n \times n}$.

Remark 2.1. For $C, D \in k^{m \times n}$, $X \in \text{GL}_m(k)$, and $Y \in \text{GL}_n(k)$ we compute

$$\text{trace}(XC Y (X^{-\top} D Y^{-\top})^\top) = \text{trace}(XC Y Y^{-1} D^\top X^{-1}) = \text{trace}(C D^\top)$$

where the second equality follows from the fact that the trace is invariant under conjugation. This shows that

$$\kappa_{X,Y}(\mathcal{C})^\perp = \kappa_{X^\top, Y^\top}(\mathcal{C}^\perp).$$

By $I_n \in k^{n \times n}$ we denote the identity matrix. The following result, which we do not need in the rest of this note, is of its own interest.

Proposition 2.2. *The rank distance preserving linear automorphisms of $k^{m \times n}$ that preserve the inner product $(-, -)$ are exactly the maps $\kappa_{X,Y}$ and, if $m = n$, $\tau_{X,Y}$ where*

$$X^\top X = aI_m, \quad YY^\top = a^{-1}I_n \text{ and } a \in k^\times, \text{ the multiplicative group of } k.$$

Proof. The maps $\kappa_{X,Y}$ and, if $m = n$, also $\tau_{X,Y}$, where $X \in \text{GL}(m, k)$ and $Y \in \text{GL}(n, k)$, are the linear automorphisms which preserve the rank distance on $k^{m \times n}$. If $n = m$ then τ_{I_n, I_n} is an improper automorphism that preserves the inner product and $\tau_{X,Y}\tau_{I_n, I_n} = \kappa_{X,Y}$, so it is enough to deal with proper automorphisms.

Suppose that $\kappa_{X,Y}$ preserves the inner product. Thus, for $A, B \in k^{m \times n}$ we have

$$\text{trace}(AB^\top) = \text{trace}(XAY(XBY)^\top) = \text{trace}((X^\top XA)(YY^\top B^\top)).$$

We put $(x_{ij}) = X^\top X$, $(y_{ij}) = YY^\top$ and denote by $E_{ij} \in k^{m \times n}$ the matrix which has a 1 at position (i, j) and 0's elsewhere. Hence, for $A = E_{ij}$ and $B = E_{lh}$, we obtain the equation

$$x_{li}y_{jh} = \text{trace}((X^\top X E_{ij})(YY^\top E_{lh})) = \text{trace}(E_{ij}E_{lh}) = \delta_{il}\delta_{jh}.$$

In particular,

$$y_{jj} = x_{ii}^{-1}$$

for all i, j . Furthermore, $\delta_{jh} = x_{ii}y_{jh}$ forces $y_{jh} = 0$ for $j \neq h$. By the same argument we get $x_{li} = 0$ for $l \neq i$. This shows $X^\top X = aI_m$ with $a \in k^\times$ and $YY^\top = a^{-1}I_n$.

Since the given maps obviously preserve both the rank distance and the inner product the proof is complete. \square

Remark 2.3. As one referee pointed out, the same proof shows that the linear rank distance preserving automorphisms that yield similarities for the inner product $(-, -)$ (which is enough to preserve the notion of duality) are the maps $\kappa_{X,Y}$ and, if $m = n$, $\tau_{X,Y}$ where $X^\top X = aI_m$, $YY^\top = bI_n$, $a, b \in k^\times$.

3 Self-dual MRD codes

Surprisingly, in characteristic 2 self-dual MRD codes in $k^{m \times n}$ do not exist. This follows immediately from the following easy, but crucial result, since a self-dual MRD code in $k^{m \times n}$ has at least minimum distance 2.

Theorem 3.1. *Assume that $\text{char}(k) = 2$ and let $\mathcal{C} \subseteq \mathcal{C}^\perp \leq k^{m \times n}$ be a self-orthogonal code. Then the all-ones matrix J is in \mathcal{C}^\perp . In particular, $d(\mathcal{C}^\perp) = 1$.*

Proof. All elements $A \in \mathcal{C}$ satisfy

$$0 = (A, A) = \sum_{i=1}^m \sum_{j=1}^n A_{ij}^2 = \left(\sum_{i=1}^m \sum_{j=1}^n A_{ij} \right)^2 = (A, J)^2$$

where J is the all-ones matrix, which is of rank 1. So $J \in \mathcal{C}^\perp$ satisfies $d(0, J) = 1$. \square

In contrast to the characteristic 2 case self-dual MRD codes may exist if $\text{char}(k)$ is odd. To see that we characterize all self-dual MRD codes \mathcal{C} in $k^{2 \times 2}$ where $k = \mathbb{F}_q$ is the finite field with q elements. Since $d(\mathcal{C}) = 2$ and $\dim(\mathcal{C}) = 2$, the projection on the first row

$$\pi : \mathcal{C} \rightarrow k^{1 \times 2}, \quad A \mapsto (a_{11}, a_{12})$$

is an isomorphism and \mathcal{C} has a unique basis of the form

$$A = \begin{pmatrix} 1 & 0 \\ a & b \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ c & d \end{pmatrix}$$

with $a, b, c, d \in k$.

Proposition 3.2. *$\mathcal{C} = \langle A, B \rangle$ is a self-dual MRD code if and only if the following two conditions hold true.*

- (i) $-1 \notin (k^\times)^2$, i.e., $q \equiv 3 \pmod{4}$.
- (ii) $a^2 + b^2 = -1$ and $(c, d) \in \{(-b, a), (b, -a)\}$.

Proof. Assume that $\mathcal{C} = \langle A, B \rangle$ is a self-dual code. Then $(A, A) = (A, B) = (B, B) = 0$ yields the equations

$$a^2 + b^2 + 1 = c^2 + d^2 + 1 = ac + bd = 0.$$

The ideal in $\mathbb{Z}[a, b, c, d]$ generated by these three polynomials contains the element

$$a^2(c^2 + d^2 + 1) - d^2(a^2 + b^2 + 1) + (bd - ac)(ac + bd) = a^2 - d^2 = (a + d)(a - d).$$

We therefore conclude that $a = \pm d$ and similarly $b = \pm c$. Thus condition (ii) is equivalent to \mathcal{C} being self-dual. Moreover \mathcal{C} is an MRD code, if all non-zero matrices in \mathcal{C} have determinant $\neq 0$, so if and only if $b \neq 0$ and

$$\det(A + xB) = \begin{cases} (x^2 + 1)b, & \text{if } (c, d) = (-b, a) \\ -(x^2 + 2\frac{a}{b}x - 1)b, & \text{if } (c, d) = (b, -a) \end{cases}$$

is an irreducible polynomial in $k[x]$. Using the fact that $a^2 + b^2 = -1$ we see that in both cases this leads to the condition that -1 is not a square in k , so $q \equiv 3 \pmod{4}$. Thus we have (i). With the same computations as above we see that the conditions in (i) and (ii) lead to a self-dual MRD code. \square

It is easy to see that all these codes are pairwise equivalent and that they are equivalent to Gabidulin codes of full length. So Proposition 3.2 may be seen as a special case of Theorem 5.10 below.

4 A criterion to be equivalent to a self-dual code

Definition 4.1. Two linear rank metric codes \mathcal{C} and $\mathcal{D} \leq k^{m \times n}$ are called properly equivalent, if there are $X \in \text{GL}_m(k)$, $Y \in \text{GL}_n(k)$ such that $\mathcal{D} = X\mathcal{C}Y$.

Note that proper equivalence is the usual notion of linear equivalence for $m \neq n$. Only for $m = n$ the proper equivalences form a subgroup of index 2 in the group of linear equivalences; the latter also includes transposition of matrices (see Section 2).

Lemma 4.2. *Let k be a finite field of odd characteristic and let $A \in k^{n \times n}$ be a symmetric matrix of full rank. Then there is a matrix $X \in \text{GL}_n(k)$ such that $A = XX^\top$ if and only if $\det(A) \in (k^\times)^2$.*

Proof. Regular quadratic forms over finite fields of odd characteristic are classified by their dimension and their determinant (see for instance [10, Chapter 2, Theorem 3.8]). In particular a quadratic form with Gram matrix $A \in \text{GL}_n(k)$ is equivalent to the standard form with Gram matrix I_n if and only if $\det(A)$ is a square. \square

Theorem 4.3. *Let k be a finite field of odd characteristic and let $\mathcal{C} \leq k^{m \times n}$ be a linear rank metric code. Then \mathcal{C} is properly equivalent to a self-dual code if and only if there are symmetric matrices $A = A^\top \in k^{m \times m}$ and $B = B^\top \in k^{n \times n}$ such that $\det(A), \det(B) \in (k^\times)^2$ are non-zero squares with*

$$\mathcal{C}^\perp = ACB.$$

Proof. Assume that there are $X \in \text{GL}_m(k)$, $Y \in \text{GL}_n(k)$ such that $\mathcal{D} := XCY = \mathcal{D}^\perp$. Then for all $C_1, C_2 \in \mathcal{C}$ we have

$$0 = \text{trace}(XC_1Y(XC_2Y)^\top) = \text{trace}(XC_1YY^\top C_2^\top X^\top) = \text{trace}(X^\top XC_1YY^\top C_2^\top)$$

Put $A := X^\top X$ and $B := YY^\top$. Then A and B are symmetric of square determinant and $\mathcal{C}^\perp = ACB$.

On the other hand assume that there are A, B as stated in the theorem. According to Lemma 4.2 there are $X \in \text{GL}_m(k)$, $Y \in \text{GL}_n(k)$ such that $A = X^\top X$, $B = YY^\top$. The same computation as above shows that XCY is a self-dual code. \square

5 Gabidulin codes in $k^{n \times n}$

We keep the assumption that $k = \mathbb{F}_q$ is a finite field, but allow $\text{char}(k)$ to be arbitrary (even or odd). Let $K := \mathbb{F}_{q^n}$ be the degree n extension field of k . For $\alpha \in K$ and $0 \leq i \leq n-1$ we define $\alpha^{[i]} := \alpha^{q^i}$ to be the image of α under the i -th iteration of the Frobenius automorphism of K/k and $\text{Trace}_{K/k}(\alpha) := \sum_{i=0}^{n-1} \alpha^{[i]}$. For a k -basis $\mathfrak{B} := (\beta_1, \dots, \beta_n)$ of K the dual basis $\mathfrak{B}^* := (\beta_1^*, \dots, \beta_n^*)$ is defined by the property that $\text{Trace}_{K/k}(\beta_i \beta_j^*) = \delta_{ij}$. If $\beta_i^* = \beta_i$ for all i , then the basis \mathfrak{B} is called a self-dual basis. Note that a dual basis always exists, but a self-dual basis exists if and only if q is even or both q and n are odd (see [5]). Let $\mathcal{T}_{\mathfrak{B}} := (\text{Trace}_{K/k}(\beta_i \beta_j))_{i,j=1,\dots,n}$ denote the Gram matrix of the trace bilinear form $(\alpha, \beta) \in K \times K \mapsto \text{Trace}_{K/k}(\alpha \beta) \in k$ with respect to the basis \mathfrak{B} . Then $\mathcal{T}_{\mathfrak{B}}$ is the base change matrix between \mathfrak{B} and its dual basis \mathfrak{B}^* , because if $\beta_i = \sum_{m=1}^n a_{mi} \beta_m^*$ with $a_{mi} \in k$ for all m , then

$$(\mathcal{T}_{\mathfrak{B}})_{ji} = \text{Trace}_{K/k}(\beta_j \beta_i) = \sum_{m=1}^n a_{mi} \text{Trace}_{K/k}(\beta_j \beta_m^*) = a_{ji}.$$

In the notation of the next definition $\mathcal{T}_{\mathfrak{B}} = \epsilon_{\mathfrak{B}^*}(\mathfrak{B})$.

Definition 5.1. Let $\mathfrak{B} = (\beta_1, \dots, \beta_n) \in K^n$ be a k -basis of K and define the map $\epsilon_{\mathfrak{B}} : K^{1 \times n} \rightarrow k^{n \times n}$ by

$$\epsilon_{\mathfrak{B}}(\alpha_1, \dots, \alpha_n) := (a_{ij}) \in k^{n \times n} \text{ if } \alpha_j = \sum_{i=1}^n a_{ij} \beta_i.$$

For $\alpha \in K$ and $\ell \in \mathbb{N}_0$ we also put $\alpha \mathfrak{B} := (\alpha \beta_1, \dots, \alpha \beta_n) \in K^n$ and $\alpha^{[\ell]} := \alpha^{(q^\ell)}$ respectively $\mathfrak{B}^{[\ell]} := (\beta_1^{[\ell]}, \dots, \beta_n^{[\ell]})$.

Lemma 5.2. *For any $\alpha \in K$ we have*

$$\epsilon_{\mathfrak{B}}(\alpha \mathfrak{B})^\top = \epsilon_{\mathfrak{B}^*}(\alpha \mathfrak{B}^*) = \mathcal{T}_{\mathfrak{B}} \epsilon_{\mathfrak{B}}(\alpha \mathfrak{B}) \mathcal{T}_{\mathfrak{B}}^{-1}.$$

Proof. Let $B := \epsilon_{\mathfrak{B}}(\alpha \mathfrak{B})$ and $\mathcal{T}_{\mathfrak{B}} =: \mathcal{T}$. If we denote the entry of a matrix A at position (i, j) by A_{ij} , then

$$\beta_j = \sum_{i=1}^n \mathcal{T}_{ij} \beta_i^*, \quad \beta_j^* = \sum_{i=1}^n (\mathcal{T}^{-1})_{ij} \beta_i, \text{ and } \alpha \beta_i = \sum_{j=1}^n B_{ji} \beta_j.$$

We compute

$$\begin{aligned} \alpha \beta_r^* &= \alpha \sum_{i=1}^n (\mathcal{T}^{-1})_{ir} \beta_i \\ &= \sum_{i=1}^n (\mathcal{T}^{-1})_{ir} (\alpha \beta_i) \\ &= \sum_{i=1}^n (\mathcal{T}^{-1})_{ir} \sum_{j=1}^n B_{ji} \beta_j \\ &= \sum_{i=1}^n (\mathcal{T}^{-1})_{ir} \sum_{j=1}^n B_{ji} \sum_{m=1}^n \mathcal{T}_{mj} \beta_m^* \\ &= \sum_{m=1}^n (\mathcal{T} B \mathcal{T}^{-1})_{m,r} \beta_m^*, \end{aligned}$$

which proves the second equality. To see the first equality let $C := \epsilon_{\mathfrak{B}^*}(\alpha^{-1} \mathfrak{B}^*)$, so $\alpha^{-1} \beta_r^* = \sum_{s=1}^n C_{sr} \beta_s^*$. Note that $BC^\top = I_n$ since

$$\begin{aligned} \delta_{ir} &= \text{Trace}_{K/k}(\alpha \beta_i \alpha^{-1} \beta_r^*) \\ &= \sum_{j=1}^n B_{ji} \sum_{s=1}^n C_{sr} \text{Trace}_{K/k}(\beta_j \beta_s^*) \\ &= \sum_{j=1}^n B_{ji} C_{jr} = (BC^\top)_{ir}. \end{aligned}$$

So $B^\top = C^{-1} = (\epsilon_{\mathfrak{B}^*}(\alpha^{-1} \mathfrak{B}^*))^{-1} = \epsilon_{\mathfrak{B}^*}(\alpha \mathfrak{B}^*)$. The last equality follows from the fact that the matrix $\epsilon_{\mathfrak{B}^*}(\alpha \mathfrak{B}^*)$ describes the k -linear map induced by the multiplication of α on K with respect to the basis \mathfrak{B}^* . \square

5.1 Automorphisms of Gabidulin codes.

In this section we determine the automorphism group of Gabidulin codes of full length $n = [K : k]$. To obtain a nice description in terms of matrices we use a normal basis $\Gamma := (\gamma, \gamma^{[1]}, \dots, \gamma^{[n-1]})$ of K over k . Define $\mathcal{T} := \mathcal{T}_\Gamma = \epsilon_{\Gamma^*}(\Gamma) \in k^{n \times n}$ to be the Gram matrix of the trace bilinear form with respect to Γ and let

$$A := \begin{pmatrix} 0 & \dots & 0 & 1 \\ 1 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ 0 & \dots & 1 & 0 \end{pmatrix} = \epsilon_\Gamma(\Gamma^{[1]}).$$

By direct matrix computations we obtain the following elementary properties of these matrices.

Remark 5.3. (i) $A\mathcal{T} = \mathcal{T}A$.

(ii) For $1 \leq j \leq n-1$ we have $\Gamma^{[j]} = \Gamma A^j$.

Definition 5.4. For $1 \leq \ell \leq n$ the Gabidulin code $\mathcal{G}_{\ell, \Gamma} \leq k^{n \times n}$ is the k -linear code

$$\mathcal{G}_{\ell, \Gamma} = \langle \epsilon_\Gamma(\gamma_i \Gamma^{[j]}) \mid 1 \leq i \leq n, 0 \leq j \leq \ell-1 \rangle.$$

Let $\mathcal{K} := \mathcal{G}_{1, \Gamma}$.

Lemma 5.5. (i) \mathcal{K} is an n -dimensional subalgebra of $k^{n \times n}$ isomorphic to $K = \mathbb{F}_{q^n}$.

(ii) For any $B \in \mathcal{K}$ we have $ABA^{-1} = B^q$. In particular $A\mathcal{K} = \mathcal{K}A$ as a set.

(iii) The normalizer in $\text{GL}_n(k)$ of \mathcal{K}^\times is the semidirect product of \mathcal{K}^\times and the cyclic group $\langle A \rangle$ of order n .

(iv) $\text{trace}(BA^\ell) = 0$ for all $B \in \mathcal{K}$ and all $1 \leq \ell \leq n-1$.

(v) The full matrix ring

$$k^{n \times n} = \mathcal{K} \oplus \mathcal{K}A \oplus \dots \oplus \mathcal{K}A^{n-1}$$

is a cyclic algebra. So for all $X \in k^{n \times n}$ there are unique $x_i \in \mathcal{K}$ such that $X = \sum_{i=0}^{n-1} x_i A^i$.

(vi) For $\ell \geq 1$

$$\mathcal{G}_{\ell, \Gamma} = \mathcal{K} \oplus \mathcal{K}A \oplus \dots \oplus \mathcal{K}A^{\ell-1}.$$

Proof. (i) The map $K \rightarrow \mathcal{K}$, $\alpha \mapsto \epsilon_\Gamma(\alpha\Gamma)$ is an isomorphism of k -algebras.
(ii) We use the isomorphism above to write $B = \epsilon_\Gamma(\beta\Gamma)$ for some $\beta \in K$ and recall that $A_{ij} = \delta_{i,(j+1)}$. We show that $AB = B^q A$ for all $B \in \mathcal{K}$. By definition we have that

$$\beta\gamma^{[j]} = \sum_{i=0}^{n-1} B_{ij}\gamma^{[i]}, \quad (AB)_{(i+1)j} = B_{ij}, \quad \text{and} \quad (B^q A)_{ij} = (B^q)_{i(j+1)}.$$

Therefore we compute for all $j = 0, \dots, n-1$

$$\begin{aligned} \sum_{i=0}^{n-1} (B^q A)_{ij}\gamma^{[i]} &= \sum_{i=0}^{n-1} (B^q)_{i(j+1)}\gamma^{[i]} = \beta^q \gamma^{[j+1]} = (\beta\gamma^{[j]})^{[1]} = (\sum_{i=0}^{n-1} B_{ij}\gamma^{[i]})^{[1]} = \\ &= \sum_{i=0}^{n-1} B_{ij}\gamma^{[i+1]} = \sum_{i=0}^{n-1} (AB)_{(i+1)j}\gamma^{[i+1]} = \sum_{i=0}^{n-1} (AB)_{ij}\gamma^{[i]}. \end{aligned}$$

So the j -th column of $B^q A$ and AB coincide.

(iii) This is well-known and widely used in geometry and group theory, see for instance [4], Kap. II, Satz 7.3.

(iv) We consider the matrices in $K^{n \times n}$. Take any primitive element $\alpha \in K$. Then $C := \epsilon_\Gamma(\alpha\Gamma) \in \text{GL}_n(K)$ has n distinct eigenvalues $\alpha, \alpha^{[1]}, \dots, \alpha^{[n-1]}$, the roots of the minimal polynomial of α over k . In particular there is a matrix $X \in \text{GL}_n(K)$ such that $X^{-1}CX = \text{diag}(\alpha, \alpha^{[1]}, \dots, \alpha^{[n-1]})$. As $ACA^{-1} = C^q$ (by (ii)) also $(X^{-1}AX)(X^{-1}CX)(X^{-1}AX)^{-1} = (X^{-1}CX)^q$, so $X^{-1}AX$ cyclically permutes the eigenspaces of $X^{-1}CX$. More precisely there are $a_i \in K$ such that

$$(X^{-1}AX)_{ij} = \begin{cases} a_i & j = i+1 \\ 0 & \text{otherwise} \end{cases}$$

where as usual the indices are taken modulo n . Because $k[C] = \mathcal{K}$, any $B \in \mathcal{K}$ is a polynomial in C and hence $X^{-1}BX$ is a diagonal matrix. So for any $1 \leq i \leq n-1$ the matrix $X^{-1}BA^iX$ is monomial with no non zero entries on the diagonal, because it induces the fixed point free permutation $(1, 2, \dots, n)^i$ on the eigenspaces of $X^{-1}CX$. In particular its trace is 0. As the trace is invariant under conjugation we also get $\text{trace}(BA^i) = \text{trace}(X^{-1}BA^iX) = 0$.

(v) Suppose that $\sum_{i=0}^{n-1} B_i A^i = 0$ where $B_i \in \mathcal{K}$. Note that $B_i A^i = \epsilon_\Gamma(\beta_i \Gamma^{[i]})$. Thus we obtain $\epsilon_\Gamma(\sum_{i=0}^{n-1} \beta_i \Gamma^{[i]}) = 0$, hence $\sum_{i=0}^{n-1} \beta_i \Gamma^{[i]} = (0, \dots, 0)$ since ϵ_Γ is injective. By [6, Chapter 3, Lemma 3.50], the $\Gamma^{[i]}$ are linearly independent over K , hence $\beta_i = 0$ for all i . This proves that the right hand side of the equation in (v) is a direct sum. The equality follows by comparing dimensions.

(vi) This follows immediately from (v) using the definition of $\mathcal{G}_{\ell, \Gamma}$. \square

We are now ready to determine the automorphism group of $\mathcal{G}_{\ell, \Gamma}$ for all ℓ . Clearly $\mathcal{G}_{0, \Gamma} := \{0\}$ and $\mathcal{G}_{n, \Gamma} = k^{n \times n}$ are fixed by all linear equivalences

(introduced in Section 2). Also for the other Gabidulin codes there are certain obvious matrices $(X, Y) \in \text{GL}_n(k) \times \text{GL}_n(k)$, so that $X\mathcal{G}_{\ell,\Gamma}Y = \mathcal{G}_{\ell,\Gamma}$ (see for instance [8]):

For notational convenience we put $\mathcal{K}^\times := \mathcal{K} \setminus \{0\}$. Then $\mathcal{K}^\times \leq \text{GL}_n(k)$ is isomorphic to the multiplicative group K^\times of K and hence cyclic of order $q^n - 1$. Let S be any generator of $\mathcal{K}^\times = \langle S \rangle$ as a group. In group theory S is often called a Singer cycle. Clearly \mathcal{K}^\times contains the subgroup of nonzero scalar matrices

$$C_{q-1} \cong k^\times \cong k^\times I_n = \langle S^{(q^n-1)/(q-1)} \rangle = \langle SS^q S^{q^2} \dots S^{(q^{n-1})} \rangle \leq \mathcal{K}^\times.$$

Furthermore if $X \in \mathcal{K}^\times$, then $X\mathcal{G}_{\ell,\Gamma} = \mathcal{G}_{\ell,\Gamma}$ and $\mathcal{G}_{\ell,\Gamma}X = \mathcal{G}_{\ell,\Gamma}$. By Lemma 5.5 (ii) conjugation by A preserves the set \mathcal{K} , so $A^j\mathcal{G}_{\ell,\Gamma}A^{-j} = \mathcal{G}_{\ell,\Gamma}$ for $j = 0, \dots, n-1$.

The next theorem shows that these obvious automorphisms already generate the full automorphism group of the Gabidulin codes.

Theorem 5.6. *For $0 < \ell < n$ the group of proper automorphisms of $\mathcal{G}_{\ell,\Gamma}$ is*

$$\text{Aut}^{(p)}(\mathcal{G}_{\ell,\Gamma}) = \{\kappa_{X,Y} \mid (X, Y) \in (A^j\mathcal{K}^\times \times A^{-j}\mathcal{K}^\times), 0 \leq j \leq n-1\}$$

which is isomorphic to the semidirect product of $C_n \cong \text{Gal}(K/k)$ with the normal subgroup $\mathcal{K}^\times Y \mathcal{K}^\times$ the central product of K^\times with itself amalgamated over k^\times .

Proof. The inclusion \supseteq is clear. To see the converse we suppose that $X\mathcal{G}_{\ell,\Gamma}Y = \mathcal{G}_{\ell,\Gamma}$ for $X, Y \in \text{GL}_n(k)$.

Claim 1: If $\mathcal{Z} := \{Z \in \text{GL}_n(k) \mid Z\mathcal{G}_{\ell,\Gamma} = \mathcal{G}_{\ell,\Gamma}\}$ then $\mathcal{Z} = \mathcal{K}^\times$:

According to Lemma 5.5 (v) we may write $Z := \sum_{i=0}^{n-1} z_i A^i \in \mathcal{Z}$ where $z_i \in \mathcal{K}$ for all i . As $I_n \in \mathcal{K}^\times \subseteq \mathcal{G}_{\ell,\Gamma}$ also $Z = ZI_n \in \mathcal{G}_{\ell,\Gamma}$, so $z_i = 0$ for $i = \ell, \dots, n-1$. If $\ell \geq 1$, then also $A \in \mathcal{G}_{\ell,\Gamma}$. Thus $ZA = \sum_{i=0}^{\ell-1} z_i A^{i+1} \in \mathcal{G}_{\ell,\Gamma}$, which implies that $z_{\ell-1} = 0$. Repeating this argument several times we obtain $z_1 = \dots = z_{n-1} = 0$ and $Z = z_0 \in \mathcal{K}$.

Claim 2: $X\mathcal{Z}X^{-1} = \mathcal{Z} (= \mathcal{K}^\times)$:

$X\mathcal{G}_{\ell,\Gamma}Y = \mathcal{G}_{\ell,\Gamma}$ is obviously invariant under left multiplication with $X\mathcal{Z}X^{-1}$. Thus Claim 1 implies $X\mathcal{Z}X^{-1} = \mathcal{Z}$.

Final step: By Claim 2 we know that $X \in \text{GL}_n(k)$ lies in the normalizer of \mathcal{K}^\times . Note that A induces by conjugation on \mathcal{K}^\times the Galois automorphism $x \mapsto x^q$ (cf. Lemma 5.5 (ii)). By Lemma 5.5 (iii) the normalizer of \mathcal{K}^\times is $N_{\text{GL}_n(k)}(\mathcal{K}^\times) = \langle A \rangle \mathcal{K}^\times$. Therefore there is some $0 \leq j \leq n-1$ such that $X \in A^j \mathcal{K}^\times$. In particular $X\mathcal{G}_{\ell,\Gamma}X^{-1} = \mathcal{G}_{\ell,\Gamma}$ and hence $\mathcal{G}_{\ell,\Gamma}XY = \mathcal{G}_{\ell,\Gamma}$. Similar to the proof of Claim 1 we conclude that $XY \in \mathcal{K}^\times$, hence $Y \in \mathcal{K}^\times A^{-j} \mathcal{K}^\times = A^{-j} \mathcal{K}^\times$. \square

Corollary 5.7. *For $0 < \ell < n$ the full automorphism group of $\mathcal{G}_{\ell, \Gamma}$ is*

$$\text{Aut}(\mathcal{G}_{\ell, \Gamma}) = \langle \text{Aut}^{(p)}(\mathcal{G}_{\ell, \Gamma}), \tau_{\mathcal{T}^{-1}, \mathcal{T} A^{\ell-1}} \rangle$$

and contains the group of proper automorphisms from Theorem 5.6 of index 2. In particular

$$|\text{Aut}(\mathcal{G}_{\ell, \Gamma})| = 2n(q^n - 1) \frac{q^n - 1}{q - 1}.$$

Proof. For any subgroup $U \leq G$ of some finite group G and a normal subgroup $N \trianglelefteq G$, we have $|U/(N \cap U)| \leq |G/N|$. So in particular the index of $\text{Aut}^{(p)}(\mathcal{G}_{\ell, \Gamma})$ in the full automorphism group is either 1 or 2 and it suffices to show that $\tau_{\mathcal{T}^{-1}, \mathcal{T} A^{\ell-1}}(\mathcal{G}_{\ell, \Gamma}) = \mathcal{G}_{\ell, \Gamma}$. To this aim let $C \in \mathcal{K}$ and $0 \leq j \leq \ell - 1$. Then

$$\begin{aligned} \tau_{\mathcal{T}^{-1}, \mathcal{T} A^{\ell-1}}(CA^j) &= \mathcal{T}^{-1}(CA^j)^{\top} \mathcal{T} A^{\ell-1} = \mathcal{T}^{-1}(C')^{\top} A^{-j} \mathcal{T} A^{\ell-1} \\ &= \mathcal{T}^{-1}(C')^{\top} \mathcal{T} A^{\ell-1-j} \end{aligned}$$

for some $C' \in \mathcal{K}$, because conjugation by A preserves \mathcal{K} as a set. The last equality follows from Remark 5.3 (i). By Lemma 5.2, we have $\mathcal{T}^{-1}(C')^{\top} \mathcal{T} = C' \in \mathcal{K}$. So $\tau_{\mathcal{T}^{-1}, \mathcal{T} A^{\ell-1}}$ maps $\mathcal{K}A^j$ onto $\mathcal{K}A^{\ell-1-j}$ and hence preserves the code $\mathcal{G}_{\ell, \Gamma}$. \square

5.2 Self-dual Gabidulin codes.

According to Theorem 3.1 and the fact that Gabidulin codes are MRD codes, there are no self-dual Gabidulin codes in even characteristic. So in this section we assume that $k = \mathbb{F}_q$, $K := \mathbb{F}_{q^n}$ and q is odd. We keep the notation from above. In particular $\Gamma = (\gamma, \gamma^{[1]}, \dots, \gamma^{[n-1]})$ is a normal basis of K/k , $\mathcal{T} := \mathcal{T}_{\Gamma}$, $\langle S \rangle = \mathcal{K}^{\times}$, and $A := \epsilon_{\Gamma}(\Gamma^{[1]})$. If the Gabidulin code $\mathcal{G}_{\ell, \Gamma}$ is equivalent to a self-dual code then $\ell = n/2$ and n needs to be even. The following facts are elementary but crucial for the proofs of Proposition 5.9 and Theorem 5.10 below.

Lemma 5.8. *Assume that n is even. Then*

- (i) $\det(A) = -1$ and $A^{\top} = A^{-1}$.
- (ii) $ASA^{-1} = S^q$.
- (iii) $A\mathcal{T} = \mathcal{T}A$. In particular $(\mathcal{T}A^{\ell})^{\top} = A^{-\ell}\mathcal{T}$.
- (iv) $\det(S)$ is a primitive element of \mathbb{F}_q .
- (v) $\det(\mathcal{T}) \notin (\mathbb{F}_q^{\times})^2$.

(vi) $\mathcal{T} S^j$ is symmetric for all $j = 0, \dots, q^n - 1$.

(vii) $S^j \mathcal{T}^{-1}$ is symmetric for all $j = 0, \dots, q^n - 1$.

(viii) $(\mathcal{T} A^j S^i)$ is symmetric if and only if $\begin{cases} j = n/2 \text{ and } (q^{n/2} + 1) \mid i \\ \text{or} \\ j = 0 \text{ and } i \in \{1, \dots, q^n - 1\}, \end{cases}$
if and only if $S^i A^j \mathcal{T}^{-1}$ is symmetric.

Proof. (i) This is clear as A is a permutation matrix of a cycle of full length n and n is even.

(ii) This follows from Lemma 5.5 (ii).

(iii) The first statement is Remark 5.3 (i). To see the second note that A is a permutation matrix, so $A^\top = A^{-1}$ and $(\mathcal{T} A^\ell)^\top = (A^\ell)^\top \mathcal{T} = A^{-\ell} \mathcal{T}$.

(iv) Because S generates \mathcal{K} as a k -algebra the minimal polynomial of S is equal to its characteristic polynomial. Moreover it also coincides with the minimal polynomial of a primitive element $\sigma \in \mathbb{F}_{q^n}$ over \mathbb{F}_q since S is a Singer cycle. Thus the determinant of S is the product of all Galois conjugates of σ , i.e. the norm of σ ,

$$\det(S) = \sigma^{(1+q+\dots+q^{n-1})} = \sigma^{(q^n-1)/(q-1)}.$$

As $\langle \sigma \rangle = \mathbb{F}_{q^n}^\times$ the order of σ is $q^n - 1$, so the order of $\det(S)$ is $q - 1$ which proves that $\det(S)$ is a primitive element in \mathbb{F}_q . In particular $\det(S) \in \mathbb{F}_q^\times \setminus (\mathbb{F}_q^\times)^2$.

(v) By Lemma 4.2, there is a self-dual basis for K/k if and only if the determinant of the trace bilinear form is a square. According to Lempel and Seroussi [5] K/k has a self-dual basis if and only if n is odd (since q is odd). As n is assumed to be even, the determinant of \mathcal{T} is a non-square.

(vi) Lemma 5.2 with $\epsilon_{\mathfrak{B}}(\alpha \mathfrak{B}) = S$ implies that $S^\top = \mathcal{T} S \mathcal{T}^{-1}$, hence

$$(\mathcal{T} S)^\top = S^\top \mathcal{T}^\top = (\mathcal{T} S \mathcal{T}^{-1}) \mathcal{T} = \mathcal{T} S.$$

(vii) This follows from (vi) because the inverse of a symmetric matrix is again symmetric.

(viii) Using the previous results we compute

$$(\mathcal{T} A^j S^i)^\top \stackrel{(ii)}{=} (\mathcal{T} S^{q^j i} A^j)^\top \stackrel{(vi)}{=} A^{-j} \mathcal{T} S^{q^j i} \stackrel{(iii)}{=} \mathcal{T} A^{-j} S^{q^j i} \stackrel{(ii)}{=} \mathcal{T} S^i A^{-j}$$

for all $0 \leq j \leq n - 1$ and $1 \leq i \leq q^n - 1$. In particular $\mathcal{T} A^j S^i$ is symmetric if and only if $\mathcal{T} A^j S^i = \mathcal{T} S^i A^{-j}$. Dividing by \mathcal{T} and using 2) we obtain the

equivalent condition $S^{q^j i} A^j = S^i A^{-j} \in \mathcal{K} A^j \cap \mathcal{K} A^{-j}$. Now $\mathcal{K} A^r \cap \mathcal{K} A^s \neq \{0\}$ if and only if $r \equiv s \pmod n$. So we obtain that $j \equiv -j \pmod n$, i.e. either $j = 0$ and then i is arbitrary, or $j = n/2$ and $(S^i)^{q^{n/2}} = (S^i)$ (i.e. $(q^{n/2} + 1) \mid i$). The last statement follows by inverting the matrix. \square

The next proposition follows by interpreting [1, Lemma 1] and [9, Theorem 18] in our language. For convenience of the reader we give a direct elementary proof.

Proposition 5.9. $\mathcal{G}_{n/2, \Gamma}^\perp = \mathcal{T} A^{n/2} \mathcal{G}_{n/2, \Gamma} \mathcal{T}^{-1}$.

Proof. We put $\mathcal{C} := \mathcal{T} A^{n/2} \mathcal{G}_{n/2, \Gamma} \mathcal{T}^{-1}$. As

$$\dim(\mathcal{C}) + \dim(\mathcal{G}_{n/2, \Gamma}) = 2 \frac{n}{2} n = n^2 = \dim(k^{n \times n})$$

it suffices to show that $\mathcal{C} \subseteq \mathcal{G}_{n/2, \Gamma}^\perp$. To see this recall that $\mathcal{G}_{n/2, \Gamma} = \bigoplus_{i=0}^{n/2-1} \mathcal{K} A^i$ where $\mathcal{K} = \{0\} \cup \{S^\ell \mid 0 \leq \ell \leq q^n - 1\}$ and $\mathcal{K} A = A \mathcal{K}$. So it is enough to show that for $i \neq j$ with $i, j \in \{0, \dots, n-1\}$ and all $m, \ell \in \{0, \dots, q^n - 1\}$

$$\text{trace}(S^m A^i (\mathcal{T} S^\ell A^j \mathcal{T}^{-1})^\top) = 0.$$

Applying Lemma 5.8 (where the relevant parts are indicated above the equalities) we compute

$$S^m A^i (\mathcal{T} S^\ell A^j \mathcal{T}^{-1})^\top \stackrel{(i), (vi)}{=} S^m A^i \mathcal{T}^{-1} A^{-j} \mathcal{T} S^\ell \stackrel{(iii)}{=} S^m A^{i-j} S^\ell \in \mathcal{K} A^{i-j}$$

where the last inclusion follows from Lemma 5.8 (ii). If $i - j$ is not divisible by n , then Lemma 5.5 (iii) tells us that all matrices in $\mathcal{K} A^{i-j}$ have trace 0. \square

In particular $\mathcal{G}_{n/2, \Gamma}$ is always equivalent to its dual code. We now apply Theorem 4.3 to obtain a criterion, when $\mathcal{G}_{n/2, \Gamma}$ is equivalent to a self-dual code.

Theorem 5.10. $\mathcal{G}_{n/2, \Gamma}$ is equivalent to a self-dual MRD code if and only if $n \equiv 2 \pmod 4$ and $q \equiv 3 \pmod 4$.

Proof. Let $\delta := \det(S)$. Then, by Lemma 5.8 (iv), $\delta \notin (k^\times)^2$.

By Proposition 5.9, we have

$$\mathcal{G}_{n/2, \Gamma}^\perp = \mathcal{T} A^{n/2} \mathcal{G}_{n/2, \Gamma} \mathcal{T}^{-1}.$$

From Theorem 5.6 we hence obtain the set of proper equivalences between $\mathcal{G}_{n/2, \Gamma}$ and $\mathcal{G}_{n/2, \Gamma}^\perp$ as

$$\{\kappa_{\mathcal{T} A^{n/2} A^j S^i, S^h A^{-j} \mathcal{T}^{-1}} \mid i, h \in \{0, \dots, q^n - 1\}, j \in \{0, \dots, n-1\}\}.$$

According to Corollary 5.7 all Gabidulin codes have improper automorphisms. So if $\mathcal{G}_{n/2,\Gamma}$ is equivalent to a self-dual MRD code, then it is properly equivalent to a self-dual MRD code.

To use Theorem 4.3 we hence need to decide for which triples (i, h, j) both matrices

$$X_{i,j} := \mathcal{T} A^{n/2} A^j S^i \text{ and } Y_{h,j} := S^h A^{-j} \mathcal{T}^{-1}$$

are symmetric and of square determinant.

By Lemma 5.8 (v) (note that we assume that n is even), $\det(X_{i,j}) \in (k^\times)^2$ if and only if $(-1)^{\frac{n}{2}+j} \delta^i \notin (k^\times)^2$ and $\det(Y_{h,j}) \in (k^\times)^2$ if and only if $(-1)^j \delta^h \notin (k^\times)^2$.

By Lemma 5.8 (viii), the matrix $X_{i,j}$ is symmetric if and only if either $j = 0$ and $(q^{n/2} + 1) \mid i$ or $j = n/2$ and i is arbitrary. The matrix $Y_{h,j}$ is symmetric if and only if either $j = 0$ and h is arbitrary or $j = n/2$ and h is a multiple of $(q^{n/2} + 1)$.

So in particular $X_{i,j}$ and $Y_{h,j}$ are symmetric of square determinant if and only if either

$$(a) \quad (-1)^{n/2} \notin (k^\times)^2 \text{ and } j = 0, (q^{n/2} + 1) \mid i, \text{ and } h \text{ is odd}$$

or

$$(b) \quad (-1)^{n/2} \notin (k^\times)^2 \text{ and } j = n/2, (q^{n/2} + 1) \mid h, \text{ and } i \text{ is odd.}$$

These conditions can be satisfied if and only if $n \equiv 2 \pmod{4}$ and $q \equiv 3 \pmod{4}$. \square

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